DYNAMIC PROGRAMMING AND THE
VARIATION OF GREEN'S FUNCTIONS

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SUMMARY

The functional equation technique of dynamic programming is applied to the study of quadratic functionals whose Euler variational equations are linear self-adjoint partial differential equations of the second order. A first consequence is the classical Hadamard variational formula for the Green's function of a region. Some extensions are indicated.

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1. INTRODUCTION

In an earlier paper [1], the functional equation technique of dynamic programming was applied to obtain a variational equation for a Green's function corresponding to a second order ordinary differential equation. In the present paper, this method is extended to apply to elliptic partial differential operators, and a first consequence is the classical Hadamard variational formula. Further results require a more high powered argumentation which we shall present subsequently.

The technique presented here utilizes the principle of optimality (see [2]) in the following fashion. Given a one-parameter family of regions, monotone under inclusion, one takes the minimum value of a certain integral on any given region R, subject to certain restrictions, to be functional of those restrictions and the region R. Then if $R^* \subset R$ the functional on R can be approximated by means of a related functional on R^* satisfying slightly different restrictions. This leads to a Gâteaux difference equation from which one easily derives the Hadamard relation.

The method is initially presented for the Laplace operator on R, and appropriate generalizations are indicated in §6 and §7.

2. PRELIMINARIES

Let R be a bounded connected region of n-dimensional real euclidean space, whose boundary ∂R is of class C_2. For convenience we shall not explicitly write out the differentials of volume and surface area in integrals over R and ∂R. Given any twice-differentiable function u on R, let Δu and u_ρ represent the Laplacian of u and the restriction of u to its limiting values on ∂R, respectively. Then if v and w are suitable functions on R and ∂R, the boundary value problem

$$(1) \qquad \Delta u = v, \quad u_\rho = w$$

possesses the unique solution

$$(2) \qquad u(p) = \int_{q \in R} g(p,q)v(q) + \int_{q \in \partial R} g_n(p,q)w(q),$$

where g is the Green's function for R normalized by the condition that

$$(3) \qquad \int_{q \in S} \Delta g(p,q) = \int_{q \in \partial S} g_n(p,q) = 1.$$

Here S is any sphere with center p which lies inside R, and where g_n is the exterior normal derivative of g on ∂S. The two integrals in (2) represent the solutions $u^{(1)}$ and $u^{(2)}$ to the boundary value problems

$$(4) \qquad \Delta u = v, \quad u_\rho = 0$$

and

(5) $$\Delta u = 0, \quad u_\rho = w$$

respectively, and they are orthogonal in the sense that

(6) $$\int_R \nabla u^{(1)} \cdot \nabla u^{(2)} = \int_{\partial R} u_\rho^{(1)} u_n^{(2)} - \int_R u^{(1)} \Delta u^{(2)} = 0,$$

where $\nabla u^{(i)}$ is the gradient of $u^{(i)}$, $i = 1, 2$.

3. A MINIMUM PROBLEM

Among those functions u such that $u_\rho = 0$, the function $u^{(1)}$ maximizes the integral

$$\int_{p \in R} \int_{q \in R} g(p,q)(\Delta u - v)(p)(\Delta u - v)(q).$$

Hence, since the maximum value is zero, and since

(7) $$\int_{q \in R} g(p,q) \Delta u(q) = u(p)$$

for any function u such that $u_\rho = 0$, one obtains an extremal condition

(8) $$\min_{u | u_\rho = 0} \int_R 2uv + |\nabla u|^2 = \min_{u | u_\rho = 0} \int_R (2v - \Delta u) u = \int_{p \in R} \int_{q \in R} g(p,q) v(p) v(q),$$

with the minimizing function $u^{(1)}$. Define

(9) $$f(v,w) = \min_{u | u_\rho = w} \int_R [2uv + |\nabla u|^2]$$

so that

(10) $$f(v,0) = \int_{p\in R}\int_{q\in R} g(p,q)v(p)v(q).$$

It should be noted that the first equality in (8) fails for those u such that $u_\rho = w \neq 0$.

Suppose that $u^{(2)}$ is given as the solution of (5). Then (9) may be rewritten, by means of (6), as

$$f(v,w) = \min_{u|u_\rho=0} \int_R [2(u+u^{(2)})v + |\nabla u + \nabla u^{(2)}|^2]$$

(11) $$= \min_{u|u_\rho=0} \int_R [2uv + |\nabla u|^2] + \int_R [2u^{(2)}v + |\nabla u^{(2)}|^2]$$

$$= f(v,0) + 2\int_R u^{(2)}v + \int_R |\nabla u^{(2)}|^2.$$

In particular, writing ϵw in place of w,

(12) $$f(v,\epsilon w) = f(v,0) + 2\epsilon\int_R u^{(2)}v + \epsilon^2\int_R |\nabla u^{(2)}|^2.$$

Since $u^{(2)}$ is known explicitly in terms of w, this enables us to compute the Gâteaux difference

(13) $$f(v,\epsilon w) - f(v,0) = 2\epsilon\int\int_{\substack{p\in R\\ q\in\partial R}} g_n(p,q)v(p)w(q) + o(\epsilon).$$

4. A FUNCTIONAL EQUATION

Let φ be a non-negative function of class C_2 on ∂R, and let ∂R^* be the surface obtained from ∂R by a displacement δn along the interior normal, where $\delta n = \epsilon\varphi$. If u is

any differentiable function on R such that $u_\rho = 0$, then the restriction u_{ρ^*} of u to ∂R^* is $-u_n \delta n + o(\varepsilon)$. We extend the definition of f to the class of regions R^* with boundaries R^* by setting

$$(14) \qquad f(\varepsilon,v,w) = \min_{u|u_{\rho^*}=w} \int_{R^*} [2uv + |\nabla u|^2].$$

If $u_\rho = 0$ then $|\nabla u|^2 = u_n^2$ on ∂R, so that the n-dimensional analog of the principle of optimality implies

$$(15) \qquad f(0,v,0) = \min_{u_n} [f(\varepsilon,v,-u_n\delta n) + \int_{\partial R} \delta n\, u_n^2 + o(\varepsilon)].$$

Set $\delta f(v,w) = f(\varepsilon,v,w) - f(0,v,w)$ and note that

$$(16) \qquad \delta f(v,-u_n\delta n) = \delta f(v,0) + o(\varepsilon).$$

In this notation one may apply (13) and (15) to obtain

$$(17) \qquad \min_{u_n} [\delta f(v,0) - 2\int_{p\epsilon R}\int_{q\epsilon\partial R} g_n(p,q)v(p)\delta n(q)u_n(q) + \int_{\partial R} \delta n\, u_n^2] = o(\varepsilon).$$

The Euler variational equation of (17) is

$$(18) \qquad u_n(q) = \int_{p\epsilon R} g_n(p,q)v(p),$$

and it follows that

$$(19) \qquad \delta f(v,0) = \int_{s\epsilon\partial R}\int_{p\epsilon R}\int_{q\epsilon R} \delta n(s)g_n(p,s)g_n(q,s)v(p)v(q) + o(\varepsilon).$$

5. THE HADAMARD VARIATION

Let $g(\epsilon,p,q)$ represent the Green's function of the region R^*, and let $\delta g(p,q) = g(\epsilon,p,q) - g(0,p,q)$. We wish to derive the Hadamard relation between δg and δn. For the region R^* (10) becomes

$$(20) \qquad f(\epsilon,v,0) = \int_{p\in R^*}\int_{q\in R^*} g(\epsilon,p,q)v(p)v(q),$$

and since g vanishes and possesses a bounded normal derivative on ∂R it follows that

$$(21) \qquad \delta f(v,0) = \int_{p\in R}\int_{q\in R} \delta g(p,q)v(p)v(q) + o(\epsilon).$$

Since v is arbitrary, (19) and (21) together imply

$$(22) \qquad \delta g(p,q) = \int_{s\in\partial R} \delta n(s)g_n(p,s)g_n(q,s) + o(\epsilon),$$

which is Hadamard's relation.

The preceding derivation is valid only when $R^* \subset R$. To prove (23) in general it suffices to consider R and R^* as regions both interior to a third region $\tilde{R}$, and to consider the difference of the variation of $\tilde{R}$ to R^* and the variation of $\tilde{R}$ to R. This device is due to Hadamard and is also applied in the standard derivation of (22).

6. LAPLACE-BELTRAMI OPERATOR

The Hadamard relation remains valid if Δ is replaced by any other self-adjoint second order differential operator

which possesses a Green's function, g. Thus, for example, Δ may be replaced by an arbitrary Laplace-Beltrami operator merely by furnishing R with an appropriate Riemannian metric. In this case there is no change in the preceding derivation.

7. INHOMOGENEOUS OPERATOR

Alternatively, we may add a multiplication to obtain the operation $u \rightarrow \Delta u + \alpha(p)u$. Assuming that $\alpha(p)$ is sufficiently small, we may again consider the functional f defined by

$$f(v,w) = \min_{u|u_\rho = w} \int_R [2uv + |\nabla u|^2 - \alpha u^2]. \tag{23}$$

The appropriate orthogonality relation is now

$$\int_R [\nabla u^{(1)} \cdot \nabla u^{(2)} - \alpha u^{(1)} u^{(2)}] = \int_{\partial R} u_\rho^{(1)} u_n^{(2)} - \int_R u^{(1)} [\Delta u^{(2)} + \alpha u^{(2)}] = 0. \tag{24}$$

The remainder of the argument proceeds as before.

Since the variational formula is independent of α, one may conclude that it is valid whenever the Green's function exists.

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